Unsteady Newton-Busemann Flow Theory—Part IV: Three Dimensional

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Abstract

An unsteady Newton-Busemann theory is given for flow past three-dimensional bodies performing small-amplitude combined pitching-plunging motions. Exact formulas that are applicable to general frequencies are obtained for the stability derivatives.

Contents

Consider a body Ω of arbitrary shape performing a small-amplitude pitching motion about an axis through the center of gravity C in a uniform hypersonic flow U_∞. Simultaneously the pivot axis undergoes a plunging motion. Let $OX^1X^2X^3$ be an inertial reference frame of Cartesian coordinates, with corresponding unit vectors $\boldsymbol{E}_1$, $\boldsymbol{E}_2$, and $\boldsymbol{E}_3$, in which OX^3 is parallel to the pivot axis and orthogonal to the freestream velocity $\boldsymbol{U}_\infty$ and OX^2 is parallel to the plunging motion. Then $\boldsymbol{U}_\infty = U_\infty \ell^i \boldsymbol{E}_i$ where $\ell^1=\cos\alpha$, $\ell^2=\sin\alpha$, $\ell^3=0$, the angle of attack α being the angle that $\boldsymbol{U}_\infty$ makes with OX^1. Here repeated Latin indices denote a sum of 1-3 and Greek of 1-2.

Let $O'x^1x^2x^3$ be a body-fixed system of Cartesian coordinates with corresponding unit vectors $\boldsymbol{e}_1$, $\boldsymbol{e}_2$, and $\boldsymbol{e}_3$, in which $O'x^3$ is kept parallel to OX^3 and the center of gravity C lies on $O'x^1$ a distance of h units from O'. All lengths are scaled by the body length ℓ, velocities by U_∞, density ρ by ρ_∞, pressure p by $\rho_\infty U_\infty^2$, and time t by ℓ/U_∞. The pitching motion of the body may be represented by the displacement angle $\theta(t)$, which is the angle between $O'x^1$ and OX^1. Let (X_c^1, X_c^2, X_c^3) be the coordinates of C. Then the plunging motion may be represented by $X_c^1=X_c=\text{const}$, $X_c^2=Y_c(t)$, $X_c^3=0$. It will be assumed that $0(\theta, \dot\theta, \gamma\equiv\dot Y_c, \dot\gamma)$, etc., are all $\ll 1$ so that their quadratic terms and higher may be neglected.

If $\boldsymbol{r}=x^i\boldsymbol{e}_i$ is the position vector of a fluid particle relative to the body-fixed coordinates, then the velocity and acceleration are given by

$$\boldsymbol{v}=[\dot x^i+(\omega^i_j x^j-\delta^i_2 h)\dot\theta+\delta^i_2\gamma]\boldsymbol{e}_i \tag{1}$$

$$\boldsymbol{a}=[\ddot x^i+2\omega^i_j\dot x^j\dot\theta+(\omega^i_j x^j-\delta^i_2 h)\ddot\theta+\delta^i_2\dot\gamma]\boldsymbol{e}_i \tag{2}$$

respectively, where $\omega^2_1=+1$, $\omega^1_2=-1$, and $\omega^i_j=0$ otherwise.

Let the equations defining the body surface be given by $x^i=H^i(\xi^1,\xi^2)$, $i=1,2,3$, where $\xi^1\xi^2$ is some curvilinear coördinate system of the body surface. The corresponding vectors tangent to the surface then are $\tau_\alpha=H^i_{,\alpha}$, $\alpha=1,2$, while the normal is given by $\boldsymbol{n}=n^i\boldsymbol{e}_i$. [Note $(\,)_{,\alpha}=\partial/\partial\xi^\alpha$.]

As in Refs. 1 and 2, the surface pressure consists of two parts: the Newtonian impact pressure and the centrifugal contribution. Let $\boldsymbol{a}=a^\alpha\tau_\alpha+a^n\boldsymbol{n}$, $\boldsymbol{v}=v^\alpha\tau_\alpha+v^n\boldsymbol{n}$, etc. The Newtonian impact pressure is then given by

$$p_{\text{Newt}}=n^in^j\ell^i\ell^j-2n^in^j\omega^j_k\ell^i\ell^k\theta-2n^i\ell^i(n^j\omega^j_kH^k-n^2h)\dot\theta-2n^in^2\ell^i\gamma \tag{3}$$

For the centrifugal part we first determine the particle motion from the condition of zero tangential acceleration,

$$\ddot\xi^\alpha+\Gamma^\alpha_{\beta\gamma}\dot\xi^\beta\dot\xi^\gamma+g^{\alpha\beta}H^i_{,\beta}[2\omega^i_jH^j_{,\gamma}\dot\xi^\gamma\dot\theta+(\omega^i_jH^j-\delta^i_2h)\ddot\theta+\delta^i_2\dot\gamma]=0 \tag{4}$$

where $g_{\alpha\beta}$, $g^{\alpha\beta}$, and $\Gamma^\alpha_{\beta\gamma}$ denote components of the metric tensor, its inverse, and Christoffel symbols, respectively. For the special case of steady flow ($\dot\theta\equiv\gamma\equiv0$), this reduces to

$$\ddot\xi^\alpha+\Gamma^\alpha_{\beta\gamma}\dot\xi^\beta\dot\xi^\gamma=0 \tag{5}$$

implying that the particle trajectories are the surface geodesics. The initial conditions required to determine the solution of Eq. (4) come from the continuity of the tangential velocity across the shock, thus, $\bar v^\alpha=\bar U^\alpha$, where the overbar indicates values at an initial point $(\bar\xi^1, \bar\xi^2)$ of entry to the shock layer.

The problem to this point has been formulated in the Lagrangian method of description. However, for treatment of continuity equation we require the solution in Eulerian form. Since t appears in Eq. (4) only through θ and γ, and since θ and γ are small, a solution may be expressed as

$$\xi^\alpha=\eta^\alpha(s,\bar\xi^1,\bar\xi^2)+\sum_{j=0}^{\infty}[E^\alpha_{\theta_j}(s,\bar\xi^1,\bar\xi^2)\theta_j(t)+E^\alpha_{\gamma_j}(s,\bar\xi^1,\bar\xi^2)\gamma_j(t)] \tag{6}$$

where

$$s=\int_{\bar t}^{t}\frac{|v_r|}{u}\,\mathrm{d}t$$

v_r being the fluid velocity relative to the body (i.e., $v_r=\dot\xi^\alpha\tau_\alpha$), u the magnitude of the velocity for the corresponding steady flow, hence $u=|v_r|_{\theta=\gamma=0}$, $\bar t$ the time when the fluid particle enters the shock layer at $(\bar\xi^1,\bar\xi^2)$. In (6) $\theta_j=\mathrm{d}^j\theta/\mathrm{d}t^j$, $\gamma_j=\mathrm{d}^j\gamma/\mathrm{d}t^j$, and, at $s=0$, $\eta^\alpha=\bar\xi^\alpha$ and $E^\alpha_{\theta_j}=E^\alpha_{\gamma_j}=0$. Simultaneously, the scaled relative speed is expressed as

$$\dot s=1+\sum_{j=0}^{\infty}[F_{\theta_j}(s,\bar\xi^1,\bar\xi^2)\theta_j+F_{\gamma_j}(s,\bar\xi^1,\bar\xi^2)\gamma_j] \tag{7}$$

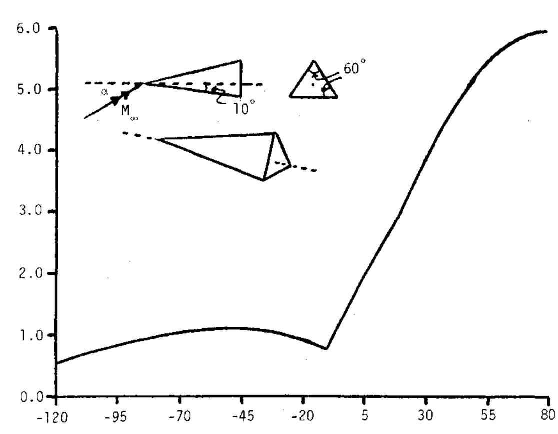

Fig. 1 Variation of damping derivative of a body with triangular cross section vs angle of attack.

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The unknown functions η^α, u, $E^\alpha_{\theta_j}$, $E^\alpha_{\gamma_j}$, F_{θ_j}, and $F_{\gamma j}$ are to be determined from Eqs. (4) together with

$$u^2 \dot{s}^2 = g_{\alpha\beta} \dot{\xi}^\alpha \dot{\xi}^\beta \tag{8}$$

Inserting Eqs. (6) and (7) into Eqs. (4) and (8) results in a system of ordinary differential equations to be solved successively. Solutions for u, $F_{\theta i}$, and $F_{\gamma j}$ are

$$u(s,\xi^1,\xi^2) = \bar{u}, \quad F_{\theta_0}(s,\xi^1,\xi^2) = \bar{F}_{\theta_0}, \quad F_{\theta_1}(s,\xi^1,\xi^2) = \bar{F}_{\theta_1} - s\bar{F}_{\theta_0}$$

$$F_{\theta_j}(s,\xi^1,\xi^2) = (-1)^j \left\{ \frac{s^j}{j!} \bar{F}_{\theta_0} - \frac{s^{j-1}}{(j-1)!} \bar{F}_{\theta_1} - \frac{1}{u^2} I^{j-1}[W_{\theta_2}] \right\}, \quad j \geq 2 \tag{9}$$

$$I[f] = \int_0^s f(\sigma,\xi^1,\xi^2)\,\mathrm{d}\sigma$$

where $\bar{u}$, $\bar{F}_{\theta_0}$ and $\bar{F}_{\theta_1}$, and $\overline{\mathrm{d}E^\alpha_{\theta_j}/\mathrm{d}s}$ are determined from initial conditions.

Thus, to determine the streamlines of the fluid flow, one must solve Eq. (5) for the geodesics η^α and then for $E^\alpha_{\theta_j}$ successively for $j = 0,1,2,\ldots$. For the important special case where $\Gamma^\alpha_{\beta\gamma,2} = 0$, which is true for many useful applications (e.g., bodies of revolution at angle of attack and any body whose sides are made up of plane surfaces such as delta wings and caret wings), Eqs. (4) are easily solved.

The continuity equation may be derived by considering a surface element of area $\mathrm{d}\bar{\Omega}$ centered at $(\bar{\xi}^1,\bar{\xi}^2)$ and the corresponding stream tube emanating from that surface element. Take a control volume $\mathrm{d}V$ to be a segment of this stream tube of length $u\mathrm{d}s$. Then, the mass of the fluid in this control volume is $\rho\mathrm{d}V$. Define λ through the relation

$$\rho \mathrm{d}V = \lambda u \mathrm{d}s \mathrm{d}\bar{\Omega} \tag{10}$$

so that $\lambda\mathrm{d}\bar{\Omega}$ is the linear density of the fluid along the stream tube. The continuity equation then reads

$$(\partial/\partial t)(\lambda \mathrm{d}\bar{\Omega}) + (\partial/\partial s)(\dot{s}\lambda \mathrm{d}\bar{\Omega}) = 0 \tag{11}$$

The boundary condition for Eq. (11), obtained by applying the law of conservation of mass across the shock, is

$$\bar{u\lambda\dot{s}} = -(\bar{U}^n - \bar{v}^n) \tag{12}$$

A solution to Eqs. (11) and (12) is of the form

$$\lambda = G(s,\xi^1,\xi^2) + \sum_{j=0}^{\infty} [K_{\theta_j}(s,\xi^1,\xi^2)\theta_j + K_{\gamma_j}(s,\xi^1,\xi^2)\gamma_j] \tag{13}$$

where G and K are determined successively by substituting Eq. (13) into Eq. (11).

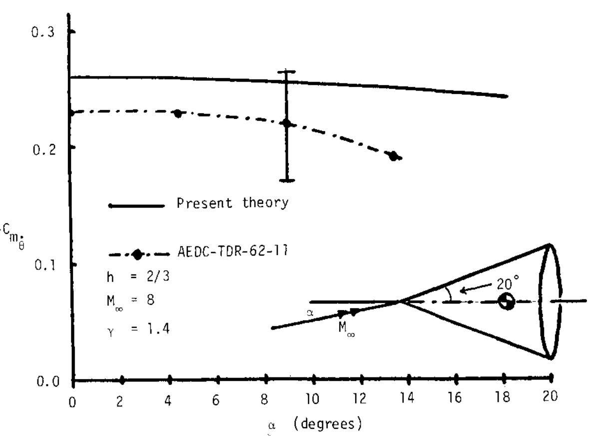

Fig. 2 Variation of damping derivative of a 20 deg cone vs angle of attack.

The stability derivatives of a three-dimensional body performing a combined pitching-plunging motion are the sum of two components: a Newtonian impact component and a centrifugal contribution. The pitching moment coefficient is defined as $C_m = M/\frac{1}{2}\rho_\infty U_\infty^2 S_b \ell$ where S_b is some "characteristic" area of the body Ω and M is the total pitching moment about C. Writing the pitching moment coefficient as

$$C_m = C_{m_0} + \sum_{j=0}^{\infty} [(-C_{m_{\theta_j}})\theta_j(t) + (-C_{m_{\gamma_j}})\gamma_j(t)] \tag{14}$$

we get, for the damping part,

$$(-C_{m\dot\theta})_{\mathrm{Newt}} = -\frac{4}{S_b}\iint_\Omega n^i \ell^i (n^j \omega^j_k H^k - n^2 h) \times [n^2(H^1 - h) - n^1 H^2]\mathrm{d}\Omega \tag{15}$$

with similar expressions for the other $(-C_{m_{\theta_j}})_{\mathrm{Newt}}$ and $(-C_{m_{\gamma_j}})_{\mathrm{Newt}}$, and

$$(-C_{m_{\theta_j}})_{\mathrm{Cent}} = \frac{2}{S_b}\int_\Omega \iint_0^{s^*} M_{\theta_j}(s,\xi^1,\xi^2)\,\mathrm{d}s\mathrm{d}\bar{\Omega}, \quad j = 0,1,2,\ldots \tag{16}$$

where $s^*(t,\xi^1,\xi^2)$ is the point where the streamline emanating from (ξ^1,ξ^2) leaves the surface and where M_{θ_j} and M_{γ_j} are determined from

$$a^n \lambda u [n^2(H^1 - h) - n^1 H^2] = \frac{S_b}{2}\left[M_0 + \sum_{j=0}^{\infty}(M_{\theta_j}\theta_j + M_{\gamma_j}\gamma_j)\right] \tag{17}$$

In Fig. 1 is plotted damping-in-pitch derivative $-C_{m\dot\theta}$ vs angle of attack α (ranging from −120 to 80 deg) of a rocket whose cross section is an equilateral triangle. In Fig. 2 is plotted the damping-in-pitch derivative $-C_{m\dot\theta}$ vs angle of attack α of a 20 deg sharp cone together with experiments[3] using air ($\gamma = 1.4$, where γ is the ratio of specific heats of the gas) at $M_\infty = 8$. Good agreement is seen between the theoretical prediction and experiments when due allowance is given to the fact that the experiments correspond to $\gamma = 1.4$. It has been shown by Hui[4] that decreasing γ tends to increase the dynamic stability. Thus, the Newton-Busemann flow theory tends to overestimate $-C_{m\dot\theta}$, typically by about 10%. The remaining difference may be attributed to flow separation for a large angle of attack that requires a special treatment and has not been accounted for in the present calculation.

As shown by Hui and Tobak,[5] the stability derivatives given here for small-amplitude oscillation contain all of the information for the large-amplitude slow oscillation of the body. On the other hand, the theory depends on the shock layer being thin as well as on the absence of stagnation points.

Acknowledgment

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